

Operation	Inverse operation
1. Interchange rows $p, q$	Interchange rows $p, q$
2. multiply row $p$ by $c \neq 0$	multiply row $p$ by $\frac{1}{c}$
3. Add $k$ times row $p$ to row $q$	Subtract $k$ times row $p$ from row $q$

## LU-Factorisation

Lower triangular matrix =  $\begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = L$

Upper triangular matrix =  $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = U$  → main diagonal

Special case diagonal matrix =  $\begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$

Let  $A$  be an  $n \times n$  matrix

$$A = LU$$

LU-factorisation: If the  $n \times n$  matrix  $A$  can be written as the product of lower triangular matrix and upper triangular matrix then  $A = LU$  is an LU-factorisation



$$\text{Ex. } A = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix}}_L \cdot \underbrace{\begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{pmatrix}}_U$$

How to find the LU-factorisation of an  $n \times n$  matrix?

### Theorem

If a square matrix can be reduced to row echelon form without using any row interchanges, then the matrix has an LU-factorisation

~~Steps~~ Ex  $A = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix}$

Steps

$$\begin{aligned} \text{① } R_2 - 2R_1 \\ R_3 - 3R_1 \end{aligned} \Rightarrow \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 12 & 16 \end{pmatrix}$$

$$\text{② } R_3 - 4R_2 \Rightarrow \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} = U$$

③ Divide each circled group by first number in the group to make it 1 (not negative 1)

$$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} = L$$



$$\text{Check: } LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} = A$$

$$\text{Ex. } \begin{pmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix}$$

$$\begin{matrix} R_2 + R_1 \\ R_3 - 2R_1 \\ R_4 + 3R_1 \end{matrix} \Rightarrow \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 10 & 4 & -9 \\ 0 & -16 & -11 & 18 \end{pmatrix} \xrightarrow{\begin{matrix} R_3 + 5R_2 \\ R_4 - 8R_2 \end{matrix}} \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

$$R_4 - 3R_3 \Rightarrow \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = U$$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{pmatrix}$$



Solve

$$3x_1 - 7x_2 - 2x_3 + 2x_4 = -9$$

$$-3x_1 + 5x_2 + x_3 = 5$$

$$6x_1 - 4x_2 - 5x_4 = 7$$

$$-9x_1 + 5x_2 - 5x_3 + 12x_4 = 11$$

$$AX=B$$

$$A = \begin{pmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix}$$

$$B = \begin{pmatrix} -9 \\ 5 \\ 7 \\ 11 \end{pmatrix}$$

① put  $A=LU$

$$\therefore LUX=B$$

② let  $y=UX$

$$\therefore LY=B \quad \text{we get } y$$

③  $y=UX$ , we get  $x$

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$UX=y$$

$$LY=B$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -9 \\ 5 \\ 7 \\ 11 \end{pmatrix}$$

$$y_1 = -9 \quad y_2 = -4 \quad y_3 = 5 \quad y_4 = 1$$



$$Y = UX$$

$$\begin{pmatrix} -9 \\ -4 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$x_4 = -1 \quad x_3 = -6 \quad x_2 = 4 \quad x_1 = 3$$

### 3. Determinants

Every square matrix can be associated with a real number called its determinant

Ex.  $A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$ ,  $\det(A) = |A| = (3 \times 4) - (2 \times 5) = 2$

\* The determinant can be +ve, -ve or zero

#### Minors and Cofactors

If  $A$  is a square matrix, then the minor  $M_{ij}$  of the element  $a_{ij}$  is the determinant of the matrix obtained by deleting the  $i$ th row and the  $j$ th column

Ex  $A = \begin{pmatrix} \textcircled{1} & 2 & 1 \\ 3 & -1 & 2 \\ \textcircled{2} & 0 & 1 \end{pmatrix}$

$$M_{11} = (-1 \times 1) - (2 \times 0) = -1$$

$$M_{31} = (2 \times 2) - (1 \times -1) = 5$$

The Cofactor  $C_{ij}$  is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$



Find the minors of

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$$

$$M = \begin{pmatrix} -3 & -5 & 4 \\ 2 & -4 & -8 \\ 5 & -3 & -6 \end{pmatrix}$$

minors

$$C = \begin{pmatrix} -1 & 5 & 4 \\ -2 & -4 & +8 \\ 5 & 3 & -6 \end{pmatrix}$$

cofactors

$$B = \begin{pmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{pmatrix}$$

$$M = \begin{pmatrix} -4 & -3 & 1 \\ -1 & -6 & 2 \\ -14 & -14 & 7 \end{pmatrix}$$

$$C = \begin{pmatrix} -4 & 3 & 1 \\ 1 & -6 & -2 \\ -14 & 14 & 7 \end{pmatrix}$$

general matrix of C

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

- If A is a square matrix, then the determinant of A is the sum of the entries in any row or column of A multiplied by their cofactors



Ex. Find the determinant of

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$$

Take row 2,  $a_{21} = 3, M_{21} = 2, C_{21} = -2$

$a_{22} = -1, M_{22} = -4, C_{22} = -4$

$a_{23} = 2, M_{23} = -8, C_{23} = 8$

$$\det(A) = (3 \times -2) + (-1 \times -4) + (2 \times 8) = 14$$

OR Take column 3,  $a_{13} = 1, M_{13} = 4, C_{13} = 4$

$a_{23} = 2, M_{23} = -8, C_{23} = -8$

$a_{33} = 1, M_{33} = -6, C_{33} = -6$

$$\det(A) = (1 \times 4) + (2 \times -8) + (1 \times -6) = 14$$

How to evaluate the determinant of a  $3 \times 3$  matrix?

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Rightarrow \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array}$$

$$\det(A) = (a_{11} \times a_{22} \times a_{33}) + (a_{12} \times a_{23} \times a_{31}) + (a_{13} \times a_{21} \times a_{32}) - (a_{31} \times a_{22} \times a_{13}) - (a_{32} \times a_{23} \times a_{11}) - (a_{33} \times a_{21} \times a_{12})$$

Ex find determinant of

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{pmatrix}$$

$$\det(A) =$$

$$(0 \times -1 \times 1) + (2 \times 2 \times 4) + (1 \times 3 \times -4) - (4 \times -1 \times 1) - (-4 \times 2 \times 0) - (1 \times 3 \times 2) = 0 + 16 - 12 + 4 - 0 - 6 = 2$$



### Special Case

If  $A$  is a triangular matrix, then its determinant is the product of the entries on the main diagonal

$$\text{Ex. } A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{pmatrix}$$

$$\det(A) = 2 \times -2 \times 1 \times 3 = -12$$